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MAPPINGS OF ALMOST HERMITIAN MANIFOLDS

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1. Introduction. The concept of a mapping of bounded dilatation recently introduced [4] is more general and natural than that of a quasiconformal mapping. Let M and N be Riemannian manifolds, and let f: $M \to N$ be a mapping of bounded dilatation of order K. When f is also harmonic, the principal result in [4], namely, Theorem 5.1, may be extended to complete manifolds M with nonpositive sectional curvature. (Theorem 5.1 says, in particular, that for an open m-ball B^m with the Poincaré metric and an n-dimensional Riemannian manifold N whose sectional curvatures are bounded above by a negative constant, if f: $B^m \to N$ is a harmonic mapping of bounded dilatation, then f is distance-decreasing up to a constant.) However, these generalizations are concerned only with the Riemannian structures of M and N as C^∞ manifolds. When these give rise to more rigid structures, e.g., when both M and N are hermitian, or, more generally, almost hermitian manifolds, and f: $M \to N$ is an almost complex mapping, then it turns out that f is of bounded dilatation. In addition, if the hermitian structures are suitably restricted (see Theorem 2) in a sense to be described in §2, f is also harmonic. It is therefore of interest to ask for the almost hermitian extensions of the Schwarz-Ahlfors lemma. Typical of the results obtained is the following generalization of a theorem due to S. S. Chern [2].

Theorem 1. *Let f: $M \to N$ be an almost complex mapping of $2n$-dimensional almost hermitian manifolds. Suppose M is a complete Kaehler manifold with nonpositive sectional curvature. If the scalar curvature of $M \geqslant -S$, and the Ricci curvature of $N \leqslant -S/2n$, where S is a positive constant, then f is volume-decreasing.*

Note that the sectional curvatures of a manifold of constant negative holomorphic curvature c lie between c and $c/4$, and that a complete simply connected m-dimensional Kaehler manifold of constant negative holomor-

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phic sectional curvature is biholomorphic with an open ball in $\mathbf{C}^m$. This is the case dealt with in [2].

For more general domains, we have the following.

Theorem 2. *Let M be a $2m$-dimensional complete almost semi-Kaehler manifold with nonpositive sectional curvature whose Ricci curvature is bounded below by a negative constant $-A$, and let N be a $2n$-dimensional quasi-Kaehler manifold whose sectional curvature is bounded above by a negative constant $-B$. If f is an almost complex mapping of M into N, then* (i) *f is distance-decreasing if $B \geqslant Ak^2/2$, where $k = \min(2m, 2n)$, and* (ii) *in the equidimensional case, f is volume-decreasing provided $B \geqslant mA$.*

For almost Kaehler manifolds, we have the following.

Corollary. *Let M be as in Theorem 2, and let N be a $2n$-dimensional almost Kaehler manifold whose holomorphic bisectional curvature is bounded above by a negative constant $-2B$. If f is an almost complex mapping of M into N, then the conclusions* (i) *and* (ii) *hold.*

In §2, the canonical connection of an almost hermitian manifold is introduced, and the definitions of a quasi-Kaehler and almost semi-Kaehler manifold are given. In §3, a formula for the Laplacian of the ratio of volume elements of M and N in the equidimensional case is derived which resembles that obtained in [2] for hermitian manifolds. The proof of Theorem 1 is given in §§4 and 5 by a method involving a conformal deformation of the hermitian metric. In the concluding section, a distortion theorem is given when the domain is not necessarily a Kaehler manifold.

2. The canonical connection. Let M be a $2n$-dimensional almost hermitian manifold with (hermitian) metric g and almost complex structure J. An *hermitian connection* on M is a connection in the bundle $U(M)$ of unitary frames on M, that is, a linear connection which is both metric (g is parallel) and almost complex (J is parallel). The existence of such a connection is assured by the general theory of connections in principal bundles.

Let Γ be an hermitian connection on M, and let $\omega = (\omega_j^i)$ be its connection form on $U(M)$. We denote by $\Theta = (\Theta^i)$ and $\Omega = (\Omega_j^i)$ the corresponding torsion and curvature forms on $U(M)$. Finally, let $\theta = (\theta^i)$ be the canonical form on $U(M)$. Then the following structural equations hold:

$$d\theta = -\omega \wedge \theta + \Theta, \tag{1}$$

$$d\omega = -\omega \wedge \omega + \Omega. \tag{2}$$

Any other hermitian connection $\tilde{\Gamma}$ has a connection form $\tilde{\omega}$ related to ω by

$$\tilde{\omega}_j^i = \omega_j^i + a_{jk}^i\theta^k + b_{jk}^i\bar{\theta}^k, \quad \bar{\theta}^k = \overline{\theta^k},$$

where the a^i_{jk} and b^i_{jk} are complex-valued functions on $U(M)$, and $a^i_{jk} + \overline{b^j_{ik}} = 0$ since ω and $\bar{\omega}$ are both skew hermitian. (The summation convention is used here and in the sequel.) These functions are chosen so that $b^i_{jk}\theta^j \wedge \bar{\theta}^k$ is the part of Θ^i of bidegree (1,1). The following statement therefore follows (see also [9]).

Proposition 1. *There is a unique hermitian connection with a pure torsion form* Θ, *that is*, $\Theta_{1,1} = 0$.

This connection is called the *canonical connection* of the almost hermitian manifold M. It was introduced by S. S. Chern [1] in the hermitian (integrable) case. The property $\Theta_{1,1} = 0$ is expressible in terms of the torsion tensor T by $T(X, JY) = T(JX, Y)$ for any vector fields X and Y on M.

Proposition 2. *The torsion form of the canonical connection on M is of bidegree* (2,0) *if and only if M is hermitian.*

Proof. The almost complex structure is integrable if and only if $d \bigwedge^{1,0} \subset \bigwedge^{2,0} \oplus \bigwedge^{1,1}$, where $\bigwedge^{p,q}$ is the module of forms of bidegree (p, q) on M. Let ϕ be a form of bidegree (1,0) on $U(M)$. Then $\phi = \phi_i\theta^i$ and

$$d\phi = \left(d\phi_i - \phi_j\omega^j_i\right) \wedge \theta^i + \phi_j\Theta^j.$$

Hence $(d\phi)_{0,2} = \phi_j\Theta^j_{0,2}$, and this is zero if and only if the (0,2) part of the torsion form vanishes.

The torsion forms are closely related to the exterior differential of the Kaehler form Φ (viewed as a tensorial form on $U(M)$). We have, using (1),

$$\Phi = i\theta^k \wedge \bar{\theta}^k, \quad i = \sqrt{-1}\,,$$

$$d\Phi = i\left(-\omega^k_j \wedge \theta^j + \Theta^k\right) \wedge \bar{\theta}^k - i\theta^k \wedge \left(-\bar{\omega}^k_j \wedge \bar{\theta}^j + \bar{\Theta}^k\right)$$

$$= -i\left(\omega^k_j + \bar{\omega}^j_k\right) \wedge \theta^j \wedge \bar{\theta}^k + i\left(\Theta^k \wedge \bar{\theta}^k - \theta^k \wedge \bar{\Theta}^k\right),$$

so that

$$d\Phi = i\left(\Theta^k \wedge \bar{\theta}^k - \bar{\Theta}^k \wedge \theta^k\right). \tag{3}$$

Separating (3) by bidegrees and recalling that $\Theta_{1,1} = \bar{\Theta}_{1,1} = 0$, we have

$$(d\Phi)_{0,3} = \overline{(d\Phi)_{3,0}} = i\Theta^k_{0,2} \wedge \bar{\theta}^k, \tag{4}$$

$$(d\Phi)_{2,1} = \overline{(d\Phi)_{1,2}} = i\Theta^k_{2,0} \wedge \bar{\theta}^k. \tag{5}$$

An almost hermitian manifold M is called *quasi-Kaehlerian* if $\bar{\partial}\Phi = (d\Phi)_{1,2}$ vanishes. (Here $\partial\psi = (d\psi)_{p+1,q}$ and $\bar{\partial}\psi = (d\psi)_{p,q+1}$ for a form ψ of bidegree (p, q)). M is called *almost semi-Kaehlerian* if Φ is co-closed. It is known (cf. [5]) that a quasi-Kaehler manifold is also almost semi-Kaehlerian.

Proposition 3. *The torsion form of the canonical connection on M is of bidegree* (0,2) *if and only if M is quasi-Kaehlerian.*

If $(d\Phi)_{0,3}$ is also zero, M is almost Kaehlerian and we can use (3) to characterize M directly.

Proposition 4. *Let Θ be the torsion form of the canonical connection on an almost hermitian manifold M, and let θ be the canonical form on $U(M)$. Then* (i) *M is almost Kaehlerian if and only if $\Theta^i \wedge \bar{\theta}^i = 0$, and* (ii) *$M$ is Kaehlerian if and only if $\Theta = 0$.*

The second part of this proposition is well known.

3. The Laplacian of the ratio of volume elements. Let M be a $2n$-dimensional almost hermitian manifold with the canonical connection of §2. For the sake of convenience, we make the discussion local by fixing a local section of $U(M)$, and pulling the various forms back to a neighborhood in M. All the formulas above still hold locally. In particular, $\{\theta^i\}$ is the coframe dual to the chosen unitary frame field. The covariant differential ∇ defined by Γ is given by

$$\nabla\theta^i = -\omega_j^i \otimes \theta^j.$$

For a complex-valued function u on M, we can write

$$\nabla u = u_i\theta^i + u_{i^*}\bar{\theta}^i,$$

where $i^* = i + n$, and

$$\begin{aligned}\nabla^2 u &= du_i \otimes \theta^i - u_i\omega_j^i \otimes \theta^j + du_{i^*} \otimes \bar{\theta}^i - u_{i^*}\bar{\omega}_j^i \otimes \bar{\theta}^j \\ &= \left(du_i - u_j\omega_i^j\right) \otimes \theta^i + \left(du_{i^*} - u_{j^*}\bar{\omega}_i^j\right) \otimes \bar{\theta}^i \\ &= \left(u_{ij}\theta^j + u_{ij^*}\bar{\theta}^j\right) \otimes \theta^i + \left(u_{i^*j}\theta^j + u_{i^*j^*}\bar{\theta}^j\right) \otimes \bar{\theta}^i \quad \text{(say)},\end{aligned}$$

where the u_{AB}, $A, B = 1, \dots, 2n$, are given by

$$\begin{aligned}u_{ij}\theta^j + u_{ij^*}\bar{\theta}^j &= du_i - u_j\omega_i^j, \\ u_{i^*j}\theta^j + u_{i^*j^*}\bar{\theta}^j &= du_{i^*} - u_{j^*}\bar{\omega}_i^j.\end{aligned}$$

Since $du = u_i\theta^i + u_{i^*}\bar{\theta}^i$, the structural equation (1) gives

$$\begin{aligned}0 &= du_i \wedge \theta^i - u_i\omega_j^i \wedge \theta^j + u_i\Theta^i + du_{i^*} \wedge \bar{\theta}^i - u_{i^*}\bar{\omega}_j^i \wedge \bar{\theta}^j + u_{i^*}\bar{\Theta}^i \\ &= \left(du_i - u_j\omega_i^j\right) \wedge \theta^i + u_i\Theta^i + \left(du_{i^*} - u_{j^*}\bar{\omega}_i^j\right) \wedge \bar{\theta}^i + u_{i^*}\bar{\Theta}^i \\ &= \left(u_{ij}\theta^j + u_{ij^*}\bar{\theta}^j\right) \wedge \theta^i + u_i\Theta^i + \left(u_{i^*j}\theta^j + u_{i^*j^*}\bar{\theta}^j\right) \wedge \bar{\theta}^i + u_{i^*}\bar{\Theta}^i.\end{aligned}$$

Comparing bidegrees we obtain

$$u_{ij^*}\bar{\theta}^j \wedge \theta^i + u_{i^*j}\theta^j \wedge \bar{\theta}^i = 0,$$

so

$$u_{ij^*} = u_{j^*i}.$$

Therefore the Laplacian of u is

$$\Delta u = g^{AB}u_{AB} = 2g^{ij^*}u_{ij^*} = 2u_{ii^*}. \tag{6}$$

Since $\partial u = (du)_{1,0} = u_i\theta^i$, and

$$\bar{\partial}\partial u = \left(d\left(u_i\theta^i\right)\right)_{1,1} = u_{ij^*}\bar{\theta}^j \wedge \theta^i,$$

the Laplacian may be computed from the components of the complex hessian of u,

$$\partial\bar{\partial}u = -\bar{\partial}\partial u = u_{ij^*}\theta^i \wedge \bar{\theta}^j. \tag{7}$$

Let N be another almost hermitian manifold of the same dimension $2n$, and let $f\colon M \to N$ be a C^∞ mapping. We fix a local unitary frame field on N, and denote by $\theta' = (\theta'^\alpha)$, $\Theta' = (\Theta'^\alpha)$, $\omega' = (\omega'^\alpha_\beta)$ and $\Omega' = (\Omega'^\alpha_\beta)$ the pullbacks by f^* of the forms corresponding to θ, Θ, ω and Ω on M. Let $\{s_\alpha\}$ be the induced unitary frame field in the induced bundle $f^{-1}T^{1,0}(N)$. Then f is *almost complex* if and only if its differential maps tangent vectors of bidegree (1,0) to tangent vector of the same bidegree. It is therefore given by

$$f_* = f_i^\alpha s_\alpha \otimes \theta^i.$$

Denoting by ∇' the covariant differential operator on $f^{-1}T^{1,0}(N)$-valued forms induced by the canonical connections in M and N, we have

$$\begin{aligned}\nabla' f_* &= s_\alpha \otimes \left(df_i^\alpha + f_i^\beta \omega'^\alpha_\beta - f_j^\alpha \omega_i^j\right) \otimes \theta^i \\ &= s_\alpha \otimes \left(f_{ij}^\alpha \theta^j + f_{ij^*}^\alpha \bar{\theta}^j\right) \otimes \theta^i \quad \text{(say)}.\end{aligned}$$

Taking the exterior derivative of $\theta'^\alpha = f_i^\alpha \theta^i$ and using (1), we obtain

$$-\omega'^\alpha_\beta \wedge \theta'^\beta + \Theta'^\alpha = df_i^\alpha \wedge \theta^i + f_i^\alpha\left(-\omega_j^i \wedge \theta^j + \Theta^i\right),$$

that is

$$\left(df_i^\alpha + f_i^\beta \omega'^\alpha_\beta - f_j^\alpha \omega_i^j\right) \wedge \theta^i + f_i^\alpha \Theta^i - \Theta'^\alpha = 0$$

from which

$$\left(f_{ij}^\alpha \theta^j + f_{ij^*}^\alpha \bar{\theta}^j\right) \wedge \theta^i + f_i^\alpha \Theta^i - \Theta'^\alpha = 0.$$

Comparing bidegrees we see that

$$f_{ij^*}^\alpha \bar{\theta}^j \wedge \theta^i = 0,$$

from which

$$f_{ij^*}^\alpha = 0. \tag{8}$$

Put $D = \det(f_i^\alpha)$, and $u = |D|^2 = D\bar{D}$. The latter is the ratio of the volume elements, $f^* V_N / V_M$. Let D_α^i denote the cofactor of f_i^α in D. Then

$$\tag{9} \begin{aligned} dD &= D_\alpha^i df_i^\alpha = D_\alpha^i\left(f_{ij}^\alpha \theta^j + f_j^\alpha \omega_i^j - f_i^\beta \omega_\beta'^\alpha\right) \\ &= D_\alpha^i f_{ij}^\alpha \theta^j + D\left(\omega_i^i - \omega_\alpha'^\alpha\right) \\ &= D_j \theta^j + D\left(\omega_i^i - \omega_\alpha'^\alpha\right) \quad \text{(say)}. \end{aligned}$$

Since ω_i^i and $\omega_\alpha'^\alpha$ are pure imaginary,

$$du = \bar{D} D_j \theta^j + D\bar{D}_j \bar{\theta}^j, \quad \partial u = \bar{D} D_j \theta^j.$$

Taking the exterior derivative of (9) and using the second structural equation (2) we obtain

$$\begin{aligned} 0 &= d\left(D_j \theta^j\right) + dD \wedge \left(\omega_i^i - \omega_\alpha'^\alpha\right) + Dd\left(\omega_i^i - \omega_\alpha'^\alpha\right) \\ &= d\left(D_j \theta^j\right) + D_j \theta^j \wedge \left(\omega_i^i - \omega_\alpha'^\alpha\right) + D\left(\Omega_i^i - \Omega_\alpha'^\alpha\right), \end{aligned}$$

so that

$$\begin{aligned} 0 &= \bar{D} d\left(D_j \theta^j\right) + D_j \theta^j \wedge \left(\bar{D}_i \bar{\theta}^i - d\bar{D}\right) + u\left(\Omega_i^i - \Omega_\alpha'^\alpha\right) \\ &= d\left(\bar{D} D_j \theta^j\right) + D_j \theta^j \wedge \bar{D}_i \bar{\theta}^i + u\left(\Omega_i^i - \Omega_\alpha'^\alpha\right). \end{aligned}$$

Hence

$$d(\partial u) = D_i \bar{D}_j \bar{\theta}^j \wedge \theta^i - u\left(\Omega_i^i - \Omega_\alpha'^\alpha\right).$$

Comparing bidegrees yields

$$\bar{\partial}\partial u = D_i \bar{D}_j \bar{\theta}^j \wedge \theta^i - u\left(\Omega_i^i - \Omega_\alpha'^\alpha\right)_{1,1}.$$

But $(\Omega_j^i)_{1,1} = R_{jkl^*}^i \theta^k \wedge \bar{\theta}^l$, where the functions R_{BCD}^A are the components of the curvature tensor. Hence

$$\left(\Omega_i^i\right)_{1,1} = R_{ikl^*}^i \theta^k \wedge \bar{\theta}^l = R_{kl^*} \theta^k \wedge \bar{\theta}^l,$$

where $R_{kl^*} X^k \bar{X}^l / g_{kl^*} X^k \bar{X}^l$ is the *Ricci curvature* in the direction of the tangent vector X. Using (7) we have

$$u_{ij^*} \bar{\theta}^j \wedge \theta^i = D_i \bar{D}_j \bar{\theta}^j \wedge \theta^i + u\left(R_{ij^*} \bar{\theta}^j \wedge \theta^i - f_i^\alpha \bar{f}_j^\beta R'_{\alpha\beta^*} \bar{\theta}^j \wedge \theta^i\right),$$

from which it follows that

$$u_{ij^*} = D_i \bar{D}_j + u\left(R_{ij^*} - f_i^\alpha \bar{f}_j^\beta R'_{\alpha\beta^*}\right).$$

Thus

$$\Delta u = 2 D_i \bar{D}_i + u\left(R - 2 f_i^\alpha \bar{f}_i^\beta R'_{\alpha\beta^*}\right),$$

where $R = 2R_{ii^*}$ is the scalar curvature of M, and

$$\tag{10} \Delta \log u = R - 2 f_i^\alpha \bar{f}_i^\beta R'_{\alpha\beta^*}$$

for $u > 0$, that is, at those points where f is locally one-to-one. In the hermitian case, this formula was obtained by Chern [2].

If the Ricci curvature of N is not greater than $-S/2n$, $S > 0$, then

$$2f_i^\alpha \bar{f}_i^\beta R'_{\alpha\beta^*} \leqslant -\frac{S}{n} f_i^\alpha \bar{f}_i^\alpha \leqslant -Su^{1/n},$$

so that

(11) $$\Delta \log u \geqslant R + Su^{1/n}.$$

4. Conformal changes of the hermitian metric. Let M be a $2n$-dimensional almost hermitian manifold with hermitian metric g. Then $\tilde{g} = e^{2\sigma}g$ is also an hermitian metric on M for any smooth real-valued function σ on M. Let $\{\theta^i\}$ be a (local) unitary coframe on (M, g). Then $\{\tilde{\theta}^i\}$, $\tilde{\theta}^i = e^\sigma\theta^i$, is a unitary coframe on $(M, \tilde{g})$. Denote by $\tilde{\theta}$, $\tilde{\omega}$, $\tilde{\Theta}$ and $\tilde{\Omega}$ the analogues for $(M, \tilde{g})$ of the forms θ, ω, Θ and Ω, respectively, on (M, g) defined in §2. Then

(12) $$\tilde{\theta} = e^\sigma\theta.$$

Hence, from (1),

$$\begin{aligned}\tilde{\Theta} &= d\tilde{\theta} + \tilde{\omega} \wedge \tilde{\theta} \\ &= e^\sigma d\sigma \wedge \theta + e^\sigma(\Theta - \omega \wedge \theta) + e^\sigma\tilde{\omega} \wedge \theta \\ &= e^\sigma\left[\Theta + (\tilde{\omega} - \omega) \wedge \theta + d\sigma \wedge \theta\right].\end{aligned}$$

Put $\tilde{\omega}_j^i - \omega_j^i = a_{jk}^i\theta^k - \bar{a}_{ik}^j\bar{\theta}^k$ and $d\sigma = \sigma_k\theta^k + \bar{\sigma}_k\bar{\theta}^k$. Then

$$e^{-\sigma}\tilde{\Theta}^i = \Theta^i + \left(a_{jk}^i\theta^k - \bar{a}_{ik}^j\bar{\theta}^k\right) \wedge \theta^j + \left(\sigma_k\theta^k + \bar{\sigma}_k\bar{\theta}^k\right) \wedge \theta^i.$$

Comparing bidegrees we see that

$$\bar{a}_{ik}^j\bar{\theta}^k \wedge \theta^j - \bar{\sigma}_k\bar{\theta}^k \wedge \theta^i = 0,$$

from which it follows that

$$a_{ik}^j = \delta_i^j\sigma_k.$$

Therefore

$$\tilde{\omega}_j^i = \omega_j^i + \delta_j^i\sigma_k\theta^k - \delta_j^i\bar{\sigma}_k\bar{\theta}^k, \; e^{-\sigma}\tilde{\Theta}^i = \Theta^i + 2\sigma_k\theta^k \wedge \theta^i.$$

Setting $d^c\sigma = i(\bar{\partial}\sigma - \partial\sigma) = i(\bar{\sigma}_k\bar{\theta}^k - \sigma_k\theta^k)$ we may write the last two formulas as

(13) $$\tilde{\omega} = \omega + \mathrm{i}d^c\sigma I,$$

(14) $$e^{-\sigma}\tilde{\Theta} = \Theta + 2\partial\sigma \wedge \theta,$$

where I is the identity matrix.

For the curvature forms, from (2) we have

$$\tilde{\Omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = d\omega + i dd^c \sigma I + \omega \wedge \omega = \Omega + i dd^c \sigma I. \tag{15}$$

Comparing bidegrees yields

$$\tilde{\Omega}_{1,1} = \Omega_{1,1} - 2\partial\bar{\partial}\sigma I, \tag{16}$$

or, in terms of components,

$$e^{2\sigma}\tilde{R}^{i}_{jkl^*} = R^{i}_{jkl^*} - 2\delta^{i}_{j}\sigma_{kl^*},$$

where $\partial\bar{\partial}\sigma = \sigma_{kl^*}\theta^k \wedge \bar{\theta}^l$. Thus, for the Ricci tensors,

$$e^{2\sigma}\tilde{R}_{kl^*} = R_{kl^*} - 2n\sigma_{kl^*},$$

and, for the scalar curvatures,

$$e^{2\sigma}\tilde{R} = R - 2n\Delta\sigma. \tag{17}$$

(The last formula is simpler than its Riemannian analogue.)

5. The volume-decreasing theorem. Let M be a complete simply connected n-dimensional Kaehler manifold of nonpositive sectional curvature. We exhaust M by a sequence of relatively compact open submanifolds $M_\rho = \{p \in M \mid \tau(p) < \rho\}$, where $\tau(p)$ is the Riemannian distance of p from a fixed point in M, that is, $M = \bigcup_{\rho<\infty} M_\rho$. Endow M_ρ with a metric $\tilde{g}$ conformally related to g, namely,

$$\tilde{g} = e^{2v_\rho}g, \quad \text{where } v_\rho = \log \frac{\rho^2}{\rho^2 - \tau^2}.$$

By (17), the scalar curvature $\tilde{R}$ of $(M_\rho, \tilde{g})$ is given by

$$\begin{aligned}\tilde{R} &= e^{-2v_\rho}(R - 2n\Delta v_\rho) \\ &= \left(\frac{\rho^2 - \tau^2}{\rho^2}\right)^2 R - \frac{4n}{\rho^4}\left[\rho^2 + \tau^2 + (\rho^2 - \tau^2)\tau\Delta\tau\right],\end{aligned}$$

where we have used the identity

$$\Delta v_\rho = \frac{dv_\rho}{d\tau}\Delta\tau + \frac{d^2 v_\rho}{d\tau^2}.$$

Suppose now the scalar curvature of M satisfies $R \geqslant -S$, where S is a positive constant. Since M has nonpositive sectional curvature, its Ricci curvature is also bounded below by $-S$. (Note that by Proposition 4, the canonical connection is the Riemannian connection.) Let $S = (2n - 1)\kappa^2$. Then (cf. [7])

$$0 < \tau\Delta\tau \leqslant (2n - 1)\kappa\tau \coth \kappa\tau < (2n - 1)\kappa\rho \coth \kappa\rho.$$

Hence

$$\tilde{R} = \left(\frac{\rho^2 - \tau^2}{\rho^2}\right) R - \varepsilon_\rho,$$

where ε_ρ is a real-valued function on M_ρ satisfying

$$0 < \varepsilon_\rho \leqslant \frac{4n}{\rho^4}\left[2\rho^2 + (2n-1)\kappa\rho^3 \coth \kappa\rho\right] = 0\left(\frac{1}{\rho}\right)$$

as $\rho \to \infty$. Therefore, for every $\varepsilon > 0$, we have

$$\tilde{R} \geqslant -S - \varepsilon \tag{18}$$

on M_ρ for sufficiently large ρ.

Let f be as in Theorem 1, and let $\tilde{f}$: $M_\rho \to N$ be its restriction to M_ρ. Consider the ratio of volume elements

$$\tilde{u} = \tilde{f}^* V_N / V_{M_\rho} = e^{-2n\sigma_\rho} u = \left(\frac{\rho^2 - \tau^2}{\rho^2}\right)^{2n} u.$$

Since the function $\tilde{u}$ is nonnegative and continuous on the closure of M_ρ, and zero on its boundary, it attains its maximum on M_ρ. If the Ricci curvature of N is not greater than $-S/2n$, then, by (11) and (18),

$$\tilde{\Delta} \log \tilde{u} \geqslant \tilde{R} + S\tilde{u}^{1/n} \geqslant S(\tilde{u}^{1/n} - 1) - \varepsilon.$$

At the maximum point x of $\tilde{u}$, $\tilde{\Delta} \log \tilde{u} \leqslant 0$, unless $\tilde{u}$ is totally degenerate. Hence $\tilde{u}(x) \leqslant (1 + \varepsilon/S)^n$. Since this inequality obviously holds at all points p of M_ρ,

$$u(p) = \left(\frac{\rho^2}{\rho^2 - \tau^2}\right)^{2n} \tilde{u}(p) \leqslant \left(\frac{\rho^2}{\rho^2 - \tau^2}\right)^{2n} \left(1 + \frac{\varepsilon}{S}\right)^n.$$

Finally, letting $\rho \to \infty$, and $\varepsilon \to 0$, we conclude that $u \leqslant 1$ thereby completing the proof of Theorem 1.

Corollary 1. *Let M be the open unit ball in $\mathbf{C}^m$ with the Poincaré-Bergman metric, and let N be an almost hermitian manifold of the same dimension. If the Ricci curvature of N is not greater than* $-2(m + 1)$, *then every almost complex mapping f: $M \to N$ is volume-decreasing.*

Corollary 2. *Let M be a symmetric bounded domain with the Bergman metric, and let N be an almost hermitian manifold of the same dimension. If the Ricci curvature of N is not greater than* -1, *then every almost complex mapping f: $M \to N$ is volume-decreasing.*

In both corollaries, M is an Einstein-Kaehler manifold with Ricci tensor $-2(m + 1)g$ and $-g$ respectively.

6. Mappings of bounded dilatation. Let M and N be C^∞ Riemannian manifolds of dimensions m and n respectively, and let g and g^* denote their respective Riemannian metrics. Let $f\colon M \to N$ be a C^∞ mapping, and denote by $\lambda_1(p) \geq \lambda_2(p) \geq \cdots \geq \lambda_m(p) \geq 0$ the eigenvalues of ${}^tf_* f_*\colon T_pM \to T_pM$, where ${}^tf_*$ denotes the transpose of the mapping f_*. If there is a positive number K such that for every $p \in M$, $\lambda_2(p) \leq \lambda_1(p) \leq K^2\lambda_2(p)$, then f is said to be of *bounded dilatation of order K*. This notion is more general and natural than that of a K-quasiconformal mapping.

The norm $\|A\|$ of a linear mapping: $A\colon V \to W$ of Euclidean vector spaces is defined by $\|A\|^2 = \operatorname{trace} {}^tAA$. If $r \leq \min(m, n)$, A may be extended to the linear mapping $\bigwedge^r A\colon \bigwedge^r V \to \bigwedge^r W$ given by $\bigwedge^r A(v_1 \wedge \cdots \wedge v_r) = Av_1 \wedge \cdots \wedge Av_r$, where the $v_i \in V$. Then

$$\|\textstyle\bigwedge^r f_*\|^2 = \sum_{1 \leq i_1 < \cdots < i_r \leq m} \lambda_{i_1} \cdots \lambda_{i_r}; \tag{19}$$

see [4]. Observe that $\|\bigwedge^r f_*\|$ bounds the ratio of r-dimensional volume elements. In particular, for any $X \in T_pM$,

$$\begin{aligned}(f^*g^*)(X, X) &= g^*(f_*X, f_*X) = g({}^tf_*f_*X, X)\\ &= \sum_{i=1}^{m} \lambda_i(\omega_i(X))^2 \leq \lambda_1 g(X, X) \leq \|f_*\|^2 g(X, X),\end{aligned}$$

where $\{\omega_i\}$, $i = 1, \ldots, m$, is the basis of covectors dual to an orthonormal basis of eigenvectors of ${}^tf_*f_*$. Thus $f^*(ds_N^2) \leq \|f_*\|^2\, ds_M^2$, where ds_M and ds_N are the distance elements defined by g and g^*, respectively.

Let $k = \min(m, n)$. Then $\operatorname{rank} f_* \leq k$. Hence, by (19),

$$\left\{\|\textstyle\bigwedge^q f_*\|^2 \Big/ \binom{k}{q}\right\}^{1/q} \geq \left\{\|\textstyle\bigwedge^r f_*\|^2 \Big/ \binom{k}{r}\right\}^{1/r}, \quad 1 \leq q \leq r \leq k, \tag{20}$$

since $\|\bigwedge^q f_*\|^2$ is the qth elementary symmetric function of $\lambda_1, \ldots, \lambda_k$.

When f is of bounded dilatation of order K, there is an inequality in the opposite direction, namely,

$$\|f_*\|^2 \leq kK\|\textstyle\bigwedge^2 f_*\|. \tag{21}$$

To see this, assume $f_* \neq 0$. Then

$$\frac{\|f_*\|^2}{\|\bigwedge^2 f_*\|} = \frac{\Sigma\,\lambda_i}{(\Sigma_{i<j}\,\lambda_i\lambda_j)^{1/2}} \leq \frac{k\lambda_1}{(\lambda_1\lambda_2)^{1/2}} = k\left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \leq kK.$$

Conversely, (21) implies that f is of bounded dilatation of some order. For,

$$\frac{\|f_*\|^2}{\|\wedge^2 f_*\|} = \frac{\Sigma\lambda_i}{(\Sigma_{i<j}\lambda_i\lambda_j)^{1/2}} \geqslant \frac{\lambda_1}{\left[\binom{k}{2}\lambda_1\lambda_2\right]^{1/2}} = \left[\frac{\lambda_1}{\lambda_2}\Big/\binom{k}{2}\right]^{1/2},$$

from which we have

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \leqslant \binom{k}{2}^{1/2}\frac{\|f_*\|^2}{\|\wedge^2 f_*\|} \leqslant k\binom{k}{2}^{1/2}K.$$

When M and N are almost hermitian manifolds, and f: $M \to N$ is an almost complex mapping, ${}^t f_* f_*$ commutes with the almost complex structure J of M. This implies that if X is an eigenvector of ${}^t f_* f_*$, then so is JX. Since X and JX are linearly independent, the eigenvectors of ${}^t f_* f_*$ have multiplicity 2 at least, so, in particular, $\lambda_1(p) = \lambda_2(p)$ for all $p \in M$. An important consequence of this is given by

Proposition 5. *An almost complex mapping of almost hermitian manifolds is of bounded dilatation of order* 1.

The following statement is an extension of the well-known fact that a holomorphic mapping of Kaehler manifolds is harmonic in terms of the corresponding Kaehler metrics.

Proposition 6 (*Lichnerowicz* [8]). *An almost complex mapping f: $M \to N$, where M is an almost semi-Kaehler manifold and N is quasi-Kaehlerian, is a harmonic mapping.*

Combining the last two propositions it is seen that an almost complex mapping f: $M \to N$, where M and N are almost semi-Kaehlerian and quasi-Kaehlerian, respectively, is harmonic and of bounded dilatation. It therefore belongs to the class recently investigated by one of the authors [4].

7. A distance-decreasing theorem. In what follows, the almost complex structures of M and N will be ignored. In fact, M and N will be C^∞ Riemannian manifolds of dimensions m and n respectively. Proceeding locally, orthonormal moving frames $\{\theta^i\}$ in M and $\{\theta^{*\alpha}\}$ in N are chosen. Let f: $M \to N$ be harmonic. Then the components of f_* with respect to the above frames are given by

$$f^*\theta^{*\alpha} = f_i^\alpha\theta^i.$$

Assume M is complete and simply connected (otherwise, pass to its simply connected covering), and has nonpositive sectional curvature. As in §5, we exhaust M by means of the submanifolds M_ρ with the identical conformally related metrics.

Let $\tilde{f}$ be the restriction of f to $(M_\rho, \tilde{g})$. Then it is shown in [3] that $\|\tilde{f}_*\|^2 = e^{-2v_\rho}\|f_*\|^2$ has a maximum on M_ρ. Furthermore, if the Ricci curvature of M is bounded below by a negative constant $-A$, then there exists a sequence of positive constants $\varepsilon(\rho)$, which goes to 0 as $\rho \to \infty$, such that

$$-R'_{\alpha\beta\gamma\delta}\tilde{f}_i^\alpha \tilde{f}_j^\beta \tilde{f}_i^\gamma \tilde{f}_j^\delta \leqslant \{A + \varepsilon(\rho)\}\|\tilde{f}_*\|^2 \tag{22}$$

at the maximum point x of $\|\tilde{f}_*\|^2$, where $\tilde{f}_i^\alpha = e^{-v_\rho}f_i^\alpha$, and the $R'_{\alpha\beta\gamma\delta}$ are the pullbacks by f^* of the components of the curvature tensor of N. On the other hand, if the sectional curvatures of N are bounded above by a negative constant $-B$,

$$-R'_{\alpha\beta\gamma\delta}\tilde{f}_i^\alpha \tilde{f}_j^\beta \tilde{f}_i^\gamma \tilde{f}_j^\delta \leqslant -2B\|\wedge^2 \tilde{f}_*\|^2. \tag{23}$$

Combining (22) and (23) we get, at x,

$$2B\|\wedge^2 \tilde{f}_*\| \leqslant \{A + \varepsilon(\rho)\}\|\tilde{f}_*\|^2. \tag{24}$$

If f is of bounded dilatation of order K, then from (21) and (24)

$$2B\|\tilde{f}_*\|^4 \leqslant \{A + \varepsilon(\rho)\}k^2K^2\|\tilde{f}_*\|^2$$

at x. Hence

$$\|\tilde{f}_*\|^2 \leqslant \tfrac{1}{2}k^2K^2\{A + \varepsilon(\rho)\}/B$$

everywhere in M_ρ. Since this inequality holds for every ρ and $\|\tilde{f}_*\| \to \|f_*\|$ as $\rho \to \infty$

$$\|f_*\|^2 \leqslant \tfrac{1}{2}Ak^2K^2/B.$$

Applying the inequality (20), this implies the following distortion theorem for intermediate volume elements, which is a considerable improvement of Theorem 5.1 in [4].

Proposition 7. *Let M be an m-dimensional complete Riemannian manifold with nonpositive sectional curvature and with Ricci curvature bounded below by a negative constant $-A$, and let N be an n-dimensional Riemannian manifold with sectional curvature bounded above by a negative constant $-B$. If f: $M \to N$ is a harmonic mapping of bounded dilatation of order K, then*

$$\|\wedge^r f_*\|^{2/r} \leqslant \frac{k}{2}\left(\frac{k}{r}\right)^{1/r}\frac{A}{B}K^2$$

for any r, $1 \leqslant r \leqslant k = \min(m, n)$.

Corollary. *Under the conditions of Proposition 7, (i) f is distance-decreasing if $2B \geqslant k^2AK^2$, and (ii) f is volume-decreasing if $m = n$ and $2B \geqslant mAK^2$.*

Propositions 5 and 6 yield the following

Proposition 8. *Let M be a $2m$-dimensional complete almost semi-Kaehler* *manifold with nonpositive sectional curvature and with Ricci curvature bounded below by a negative constant $-A$. Let N be a $2n$-dimensional quasi-Kaehler manifold whose sectional curvatures are bounded above by a negative constant $-B$. If f: $M \to N$ is an almost complex mapping, then*

$$\| \wedge^r f_* \|^{2/r} \leqslant \frac{k}{2} \binom{k}{r}^{1/r} \frac{A}{B}$$

for any r, $1 \leqslant r \leqslant k = \min(2m, 2n)$.

Theorem 2 is now a consequence of Proposition 8.

The corollary to Theorem 2 is obtained from the following formula:

$$K(X, Y)\|X \wedge Y\|^2 + K(X, JY)\|X \wedge JY\|^2 + K(JX, Y)\|JX \wedge Y\|^2 + K(JX, JY)\|JX \wedge JY\|^2 \leqslant 2H(X, Y)\|X\|^2\|Y\|^2,$$

valid for almost Kaehler manifolds (see [6, formula 4.5]) where $K(X, Y)$ and $H(X, Y)$ are the sectional curvature and the holomorphic bisectional curvature, respectively, determined by the tangent vectors X and Y. From this formula, it is seen that (23) also holds under the assumption that the holomorphic bisectional curvatures of N are bounded above by a negative constant $-2B$.

By taking $M = \mathbf{C}^m$ with the standard flat metric Proposition 8 yields the following generalization of Liouville's theorem as well as Picard's first theorem.

Proposition 9. *Let N be a quasi-Kaehler manifold with negative sectional curvature bounded away from zero. If f: $\mathbf{C}^m \to N$ is an almost complex mapping, then it is a constant mapping.*

We take this opportunity to correct an error in [4], from which §§6 and 7 of this paper originated. The inequality in Lemma 2.2 should be replaced by formula (21) above. (In the hypotheses preceding Lemma 2.1 the expression l_s should be replaced by l_{s-1}.) As a consequence, the factor K^4 in Theorems 4.1, 5.1 and 5.4, as well as in Corollaries 4.2, 4.3 and 5.1 can be replaced by K^2. This correction actually improves these results. Moreover, since for $m = n = 2$, the notion of a mapping of bounded dilatation of order K is identical with that of a K-quasiconformal mapping, the factor K^4 appearing in Theorem 1 of [3] may be replaced by K^2, thereby improving that statement when M and N are surfaces.

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